

STEINHAUS' LATTICE-POINT PROBLEM FOR BANACH SPACES

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ABSTRACT. Given a positive integer n , one may find a circle on the Euclidean plane surrounding exactly n points of the integer lattice. This classical geometric fact due to Steinhaus has been recently extended to Hilbert spaces by Zwoleski, who replaced the integer lattice by any infinite set which intersects every ball in at most finitely many points. We investigate the Banach spaces satisfying this property, which we call (S), and show that all strictly convex Banach spaces have (S). Nonetheless, we construct a norm in dimension three which has (S) but fails to be strictly convex. Furthermore, the problem of finding an equivalent norm enjoying (S) is studied. With the aid of measurable cardinals, we prove that there exists a Banach space having (S) but with no strictly convex renorming.

1. INTRODUCTION

It is a well-known property of the Euclidean plane, which goes back to H. Steinhaus [11], that for any $n \in \mathbb{N}$ one may find a circle surrounding exactly n points of the integer lattice. P. Zwoleński [14] generalised this fact to the setting of Hilbert spaces. He replaced the set of lattice points by a more general *quasi-finite* set, *i.e.* an infinite subset A of a metric space X such that each ball in X contains only finitely many elements of A . His result reads as follows.

Theorem 1 (Zwoleński [14]). *If A is a quasi-finite subset of a Hilbert space X , then there is a dense set $Y \subset X$ such that for every $y \in Y$ and $n \in \mathbb{N}$ there exists a ball B centred at y with $|A \cap B| = n$.*

In this note we extend the above result to a larger class of Banach spaces and give a simple geometrical characterisation of what we shall call *Steinhaus' property* of a Banach space X :

- (S) For any quasi-finite set $A \subset X$ there exists a dense set $Y \subset X$ such that for all $y \in Y$ and $n \in \mathbb{N}$ there exists a ball B centred at y with $|A \cap B| = n$.

We shall prove that (S) is equivalent to the following condition which involves only the shape of the unit ball of X :

- (S') For all $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$, and each $\delta > 0$, there exists a $z \in X$ with $\|z\| < \delta$ such that one of the vectors: $x + z$ and $y + z$ has norm greater than 1, whereas the other has norm smaller than 1.

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In other words, condition (S') means that the unit sphere of X does not look locally the same at any two different points. Using that equivalence we extend Zwoleński's result, in particular, to strictly convex Banach spaces. This motivates the question whether every Banach space satisfying (S) must be strictly convexifiable. A negative answer is obtained by an ultraproduct argument which requires the assumption that a measurable cardinal exists.

2. CHARACTERISATION OF STEINHAUS' PROPERTY

Theorem 2. *For every Banach space X properties (S) and (S') are equivalent.*

Proof. First, we prove that (S) implies (S'). So, assume (S) and fix any $\delta > 0$ and $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$. Consider any quasi-finite set $A \subset X$ such that $A \cap (1+\delta)B_X = \{x, y\}$, where B_X stands for the closed unit ball of X . According to (S) there is a $u \in X$, $\|u\| < \delta/2$, such that for some $r > 0$ the open ball $B(u, r)$ contains exactly one element of A . Suppose there is an $a \in A \setminus \{x, y\}$ belonging to $B(u, r)$. Then

$$r > \|a - u\| \geq \|a\| - \|u\| > (1 + \delta) - \frac{\delta}{2} = 1 + \frac{\delta}{2},$$

hence $\|x - u\| < r$, that is $x \in B(u, r)$; a contradiction. Consequently, $B(u, r)$ contains exactly one of the points x and y , say $x \in B(u, r)$ and $y \notin B(u, r)$. Then

$$1 - \frac{\delta}{2} < \|x - u\| < r \leq \|y - u\| < 1 + \frac{\delta}{2}.$$

Suppose that $r \leq 1$, $r = 1 - \varepsilon$ with some $\varepsilon \in [0, \delta/2)$ and take any number ρ satisfying

$$0 < \rho < \min\left\{r - \|x - u\|, \frac{\delta}{2} - \varepsilon\right\}.$$

Obviously, we may find $v \in X$ with $\|v\| \leq \varepsilon + \rho$ such that $\|y - (u + v)\| \geq r + \varepsilon + \rho > 1$. Then we also have

$$\|x - (u + v)\| \leq \|x - u\| + \|v\| < r - \rho + \|v\| \leq 1.$$

Therefore, putting $z = -(u + v)$ completes the proof of our claim, since $\|u + v\| < \varepsilon + \rho + \delta/2 < \delta$. We proceed similarly in the case where $r > 1$.

Now assume that the Banach space X satisfies (S') and let $A \subset X$ be a quasi-finite set. For any $n \in \mathbb{N}$ set

$$G_n = \{x \in X : |A \cap B(x, r)| = n \text{ for some } r > 0\}.$$

It is evident, in view of the definition of a quasi-finite set, that each G_n is an open subset of X . We shall prove that it is also dense.

Assume, in search of a contradiction, that there is an open ball $U = B(x_0, r)$ in X not intersecting G_n . Rescaling U if necessary, we may thus suppose that $A \cap U = \emptyset$. For any $t > 0$ let U_t stand for the set U scaled by t , i.e. $U_t = x_0 + t(U - x_0)$. Since $x_0 \notin G_n$, there

is the greatest non-negative integer $m < n$ such that $|A \cap U_r| = m$ with some $r > 1$. Then for every $s > r$ we have either $|A \cap U_s| = m$ or $|A \cap U_s| > n$. Set

$$s = \inf\{r > 1: |A \cap U_r| > n\}.$$

Then exactly m points $a_1, \dots, a_m \in A$ lie in the ball U_s , whereas at least two such points lie on the boundary of U_s ; let us call them b_1, \dots, b_k ($k \geq 2$). Pick any $\delta > 0$ such that

$$\{a_i: 1 \leq i \leq m\} \subset B(x_0 + u, s) \subset \{a_i, b_j: 1 \leq i \leq m, 1 \leq j \leq k\}$$

for every $u \in X$ with $\|u\| < r_0 s \delta$. Each of the vectors $(b_j - x_0)/r_0 s$ lies on the unit sphere. Applying assumption (S) for two of them (*e.g.* for $j = 1, 2$) and for δ chosen above, we get a $z \in X$ with $\|z\| < \delta$ such that one of the vectors: $b_j - x_0 - r_0 s z$ ($j = 1, 2$) has norm greater than $r_0 s$, whereas the other has norm smaller than $r_0 s$. By decreasing δ , if necessary, we may also assume that the point $x_0 + r_0 s z$ still lies in U . Therefore the ball $B(x_0 + r_0 s z, s)$ with the centre in U contains all of a_i 's ($1 \leq i \leq m$) and at least one but not all of b_j 's ($1 \leq j \leq k$). Repeating this construction finitely many times, we get a point $z_0 \in U$ such that for some $r > 0$ we have $|A \cap B(z_0, r)| = m + 1$. If $m + 1 = n$ we are done. Otherwise, we apply the same procedure for the new centre z_0 instead of x_0 until we get a point belonging to $G_n \cap U$.

By the Baire Category Theorem, the set $Y = \bigcap_{n=1}^{\infty} G_n$ is dense in X and, obviously, for each $y \in Y$ and $n \in \mathbb{N}$ there is a ball B centred at y with $|A \cap B| = n$. This completes the proof of (S). \square

3. EXAMPLES

In this section we will demonstrate some applications of Theorem 2 in concrete situations. We begin with a strengthening of Theorem 1.

Corollary 3. *Every strictly convex Banach space X satisfies (S).*

Proof. It is enough to verify condition (S'). Let $\delta > 0$ and $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$ be given. Then each point inside the segment \overline{xy} , joining x and y , has norm smaller than 1, whereas each point lying on the straight line passing through x and y , but outside \overline{xy} , has norm larger than 1. Therefore, any point $z \in X$ satisfying $0 < \|z\| < \delta$ and $x + z \in \overline{xy}$ does the job. \square

Now, we will see that strictly convex spaces do not exhaust the whole class of Banach spaces satisfying Steinhaus' condition. In fact, these two classes differ already in dimension three.

Example. There is a norm $\|\cdot\|$ in \mathbb{R}^3 such that $(\mathbb{R}^3, \|\cdot\|)$ contains ℓ_{∞}^2 isometrically (and hence is not strictly convex), nonetheless it satisfies condition (S). We shall briefly describe the idea of how to define such a norm as a Minkowski functional of a certain convex symmetric set $B \subset \mathbb{R}^3$. Let us construct the boundary of B starting with putting the square with vertices $(\pm 1, \pm 1, 0)$ onto the xy -plane, so that we would have a subspace isometric to ℓ_{∞}^2 . Next, we define the curves on the boundary of B which join the points $(x, 1, 0)$ and $(0, 0, 1)$ for every $x \in [-1, 1]$. We do it in such a way that for all $-1 \leq x_1 < x_2 \leq 1$ the

curve corresponding to x_1 is more flat at $(x_1, 1, 0)$ than the one corresponding to x_2 is at $(x_2, 1, 0)$.

For any fixed $x \in [-1, 1]$ the curve starting from $(x, 1, 0)$ lies on the plane spanned by $(x, 1, 0)$ and $(0, 0, 1)$ and, on this plane, it joins the points $(\sqrt{x^2 + 1}, 0)$ and $(0, 1)$, where the coordinates are: $t = \sqrt{x^2 + y^2}$ and z . We define that curve by the equation

$$z_\alpha(t) = 1 - \left(\frac{t}{\sqrt{x^2 + 1}} \right)^\alpha \text{ for } 0 \leq t \leq \sqrt{x^2 + 1},$$

and if the parameter α behaves appropriate as a function of $x \in [-1, 1]$, then we will get the flattening effect announced earlier. We will see in a moment what should be a suitable choice for $\alpha(x)$. In this way we define a surface over the triangle $\Delta_1 = \{(x, y) \in \mathbb{R}^2 : y \geq |x|\}$ by the formula

$$z(x, y) = 1 - \left(\frac{x^2 + y^2}{x^2 + 1} \right)^{\alpha(x)/2} \text{ for } (x, y) \in \Delta_1.$$

The rest of the square is divided into similar three triangles Δ_2 , Δ_3 and Δ_4 , arranged clockwise on the xy plane. We apply the same procedure for each of them taking care of the behaviour of the parameter α likewise we did for Δ_1 . Finally, we extend our surface below the xy -plane by reflecting it with respect to the origin. The part lying over the triangles Δ_1 and Δ_2 (for a suitable choice of the function α) is depicted in Figure 1.

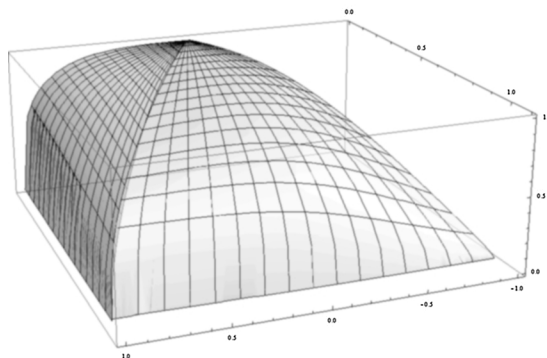


FIGURE 1. The part of the boundary of B lying above $\Delta_1 \cup \Delta_2$

Let $\|\cdot\|$ be the Minkowski functional of B . In order to show that the Banach space $(\mathbb{R}^3, \|\cdot\|)$ satisfies condition (S') it is enough to consider any two different vectors having norm 1 which lie on the same edge of the square with vertices $(\pm 1, \pm 1, 0)$, since for any other two vectors from the boundary of B we may apply the same simple argument as in the case of a strictly convex space (provided, of course, our choice of $\alpha(x)$ gives a strictly concave function $z(x, y)$). With no loss of generality we may also suppose these two vectors lie inside the segment $[-1, 1] \times \{1\} \times \{0\}$; let us call them $(x_1, 1, 0)$ and $(x_2, 1, 0)$, where $-1 < x_1 < x_2 < 1$. Fix also any $\delta > 0$ and for any $x \in [-1, 1]$ define $\xi(x)$ to be the unit vector tangent to the curve starting from $(x, 1, 0)$ and lying on the plane spanned

by $(x, 1, 0)$ and $(0, 0, 1)$, above the xy -plane. For any such x and $\alpha = \alpha(x)$ we have $z'_\alpha(\sqrt{x^2 + 1}) = -\alpha/\sqrt{x^2 + 1}$. Hence, if we take α of the form $\alpha(x) = \beta(x)\sqrt{x^2 + 1}$ with any strictly increasing map $\beta: [-1, 1] \rightarrow \mathbb{R}$ satisfying $\beta(x) > 1/\sqrt{x^2 + 1}$ for $-1 \leq x \leq 1$, then the flattening effect works. Namely, the angle between $\xi(x_1)$ and the xy -plane is smaller than the one between $\xi(x_2)$ and the xy -plane. (Notice that for any $x \in [-1, 1]$ the angle between $\xi(x)$ and the segment joining the origin with $(x, 1, 0)$ equals the angle between $\xi(x)$ and the xy -plane.) Take a vector z with $\|z\| < \delta$ which is “almost” parallel to $\xi(x_2)$ in such a way that $(x_2, 1, 0) + z$ goes inside the ball B . Let P be the foot of perpendicular of $(x_1, 1, 0) + z$ on the xy -plane and let the line passing through the origin and P meet the segment $[-1, 1] \times \{1\} \times \{0\}$ at some point $(x_0, 1, 0)$. Of course, $x_0 < x_1$ and $\xi(x_2) = (x_0 - x_1, 0, 0) + z'$ with z' lying on the plane spanned by $(x_0, 1, 0)$ and $(0, 0, 1)$. By decreasing the length of z we may get x_0 arbitrarily close to x_1 , thus the angle between z' and the xy -plane may be almost the same as the one given by z . But it is greater than the angle between $\xi(x_0)$ and the xy -plane and this in turn may be arbitrarily close to the angle given by $\xi(x_1)$. Consequently, by decreasing the length of z we may guarantee that the vector $x_1 + z$ goes outside the ball B .

Remark 4. The above example shows that there is a Banach space X satisfying (S'), but containing two such vectors $x, y \in X$, with $\|x\| = \|y\| = 1$, that for some $\delta > 0$ it is impossible to increase $\|x\|$ and decrease $\|y\|$ by adding to x and y the same vector $z \in X$ with $\|z\| < \delta$. In other words, condition (S') cannot be strengthened by claiming which one of $x + z$ and $y + z$ has norm greater than 1.

Remark 5. Of course, if X is a non-strictly convex Banach space with $\dim X = 2$, then condition (S') fails to hold. Therefore, Steinhaus' condition is equivalent to strict convexity in the class of Banach space with dimension at most 2.

The next corollary demonstrates that the classical $L_p(\mu)$ -spaces for atomless measures μ also satisfy Steinhaus' condition, giving thus another example of a non-strictly convex space with this property (the case where $p = 1$).

Corollary 6. *For every atomless measure space (Ω, Σ, μ) , and each $1 \leq p < \infty$, the space $L_p(\mu)$ satisfies (S).*

Proof. By Luther's theorem [9], there is a decomposition $\mu = \mu_1 + \mu_2$ with μ_1 being semi-finite (*i.e.* for each $A \in \Sigma$, $\mu_1(A) = \infty$, there is a subset $B \in \Sigma$ of A with $0 < \mu_1(B) < \infty$) and μ_2 being degenerate (*i.e.* the range of μ_2 is contained in $\{0, \infty\}$). The space $L_p(\mu)$ is then isometric to $L_p(\mu_1)$ (for any $f \in L_p(\mu)$ we have $\mu_2\{x: f(x) \neq 0\} = 0$, thus the identity map yields the desired isometry). Therefore, we may (and do) suppose that μ is semi-finite.

Fix two functions $f, g \in L_p(\mu)$ with $f \neq g$ and $\|f\| = \|g\| = 1$, and let $\delta > 0$ be given. Interchanging f and g , if necessary, we may assume that there is a set $F \in \Sigma$ such that $0 < \mu(F) < \infty$ and $f(x) > g(x)$ for every $x \in F$. Since

$$F = \bigcup_{n=1}^{\infty} \left\{ x \in F : f(x) > g(x) + \frac{1}{n} \right\},$$

we may also suppose that for some $\varepsilon > 0$ and all $x \in F$ we have $f(x) > g(x) + \varepsilon$. Approximating f and g by step functions we may find a measurable set $F' \subset F$ with $\mu(F') > 0$ and some $c_f, c_g \in \mathbb{R}$ such that

$$|f(x) - c_f| < \frac{\varepsilon}{5} \quad \text{and} \quad |g(x) - c_g| < \frac{\varepsilon}{5} \quad \text{for } x \in F'.$$

Hence, $c_f > c_g + 3\varepsilon/5$ and $m_f > M_g + \varepsilon/5$, where $m_f = \inf f(F')$ and $M_g = \sup g(F')$. We have three possibilities:

- (i) $m_f > 0$ and $M_g \geq 0$,
- (ii) $m_f > 0$ and $M_g < 0$,
- (iii) $m_f \leq 0$ and $M_g < 0$.

With no loss of generality suppose that either (i) or (ii) occurs (the case (iii) is analogous to (i)). Then there is a positive number d such that $|m_f - d| < m_f$ and $|M_g - d| > |M_g|$; indeed, in the former case we shall take any $d \in (2M_g, 2m_f)$, while in the latter one any arbitrarily small d does the job.

Now, observe that

$$(1) \quad |f(x) - d| < |f(x)| \quad \text{and} \quad |g(x) - d| > |g(x)| \quad \text{for } x \in F'.$$

Indeed, for the first inequality note that in the case where $f(x) \geq d > 0$ it holds trivially true, while in the opposite case we have

$$|f(x) - d| = d - f(x) \leq d - m_f \leq |d - m_f| < m_f \leq |f(x)|.$$

For the other one observe that since $M_g < d$ (recall $|M_g - d| > |M_g|$), we have $g(x) < d$, thus in the case where $g(x) \geq 0$ we have

$$|g(x) - d| = d - g(x) \geq d - M_g = |d - M_g| > |M_g| \geq g(x) = |g(x)|,$$

whereas in the case where $g(x) < 0$ this inequality is trivial.

By the Darboux property of finite atomless measures (*e.g.* see [6, §215]), there is a measurable set $H \subset F'$ with $0 < \mu(H) < \delta/|d|$. Then $\|d\mathbf{1}_H\| < \delta$, where $\mathbf{1}_H$ stands for the characteristic function of D , while inequalities (1) imply $\|f + |d|\mathbf{1}_H\| < 1$ and $\|g + |d|\mathbf{1}_H\| > 1$. This completes the proof of (S') and, consequently, also of (S). \square

Theorem 2 gives an immediate solution of Steinhaus' problem for $C(K)$ -spaces (of all scalar-valued continuous functions defined on a compact Hausdorff space K).

Corollary 7. *If K is a compact Hausdorff space with at least two points, then $C(K)$ does not have (S).*

Proof. Pick any two distinct points $u, v \in K$, and their disjoint neighbourhoods U and V . Since K is completely regular, there is a continuous map $\varphi: K \rightarrow [0, 1]$ such that $\varphi(u) = 1$ and $\varphi|_{K \setminus U} = 0$. Similarly, since $K \setminus U$ is also completely regular, there is a continuous map $\varphi_1: K \setminus U \rightarrow [0, 1/2]$ such that $\varphi_1(v) = 1/2$ and $\varphi_1|_{K \setminus (U \cup V)} = 0$. Then the mapping $\psi: K \rightarrow [0, 1]$ defined by

$$\psi(x) = \begin{cases} \varphi(x) & \text{for } x \in U, \\ \varphi_1(x) & \text{for } x \in K \setminus U, \end{cases}$$

is continuous and, of course, $\varphi \neq \psi$. So, both functions φ and ψ belong to the unit sphere of $C(K)$, but for any $\delta \in (0, 1/2)$ condition (S') is violated. \square

4. MEASURABLE CARDINALS AND NON-STRICTLY CONVEXIFIABLE SPACES HAVING (S)

Let κ be an uncountable cardinal. A non-principal filter \mathcal{U} over some set is κ -complete, if it is closed under intersections of fewer than κ sets, meaning that if $\mathcal{C} \subset \mathcal{U}$ has cardinality less than κ then $\bigcap \mathcal{C} \in \mathcal{U}$. (Filters which are ω_1 -complete are said to be σ -complete.) Thus, every κ -complete filter is closed under countable intersections and this property is clearly impossible to be satisfied by ultrafilters on ω . An uncountable cardinal κ is *measurable* if there exists a κ -complete ultrafilter on a set of cardinality κ . Every measurable must be *strongly inaccessible* (e.g. see [7, Lemma 10.4]), which means that if $\lambda < \kappa$ is a cardinal then $2^\lambda < \kappa$. Measurable cardinals 'may not' exist as shown by Scott ([10]), who proved the incompatibility of measurable cardinals with the Axiom of Constructibility ' $V = L$ '.

We shall employ measurable cardinals to prove that there exist Banach spaces having (S) but with no strictly convex renormings. To do this, we require the notion of ultraproduct of Banach spaces.

Let Γ be an infinite set and let \mathcal{U} be a non-principal ultrafilter on Γ . Suppose $(E_i)_{i \in \Gamma}$ is a family of Banach spaces and let $\ell_\infty(E_i)$ denote the ℓ_∞ -sum of this family, i.e. the Banach space of all functions $x = (x_i)$ with $x_i \in E_i$ ($i \in \Gamma$) and $\sup_{i \in \Gamma} \|x_i\| < \infty$, endowed with the supremum norm. Furthermore, let $c_{\mathcal{U}}$ be the subspace of $\ell_\infty(E_i)$ consisting of those functions x for which $\lim_{i, \mathcal{U}} \|x_i\| = 0$. The quotient

$$\prod_{i \in \Gamma}^{\mathcal{U}} E_i := \ell_\infty(E_i) / c_{\mathcal{U}}$$

is a Banach space, called *the ultraproduct* of the family $(E_i)_{i \in \Gamma}$ (or, *the ultrapower*, when the spaces E_i are all equal to some space E ; in this case we denote the ultrapower by $E_{\mathcal{U}}$). The norm of an element $\mathbf{x} = [(x_i)] \in \prod_{i \in \Gamma}^{\mathcal{U}} E_i$ can be calculated accordingly to the formula $\|\mathbf{x}\| = \lim_{i, \mathcal{U}} \|x_i\|$ as it does not depend on the choice of a representative. We refer to [8], [12] and [13] for the detailed study of ultraproducts in the Banach space theory.

Proposition 8. *Assume there exists a measurable cardinal. Let \mathcal{U} be a σ -complete, non-principal ultrafilter. Suppose a Banach space E has (S). Then $E_{\mathcal{U}}$ has (S) as well.*

Proof. Let $\mathbf{x} = [(x_i)]$ and $\mathbf{y} = [(y_i)]$ be unit vectors in $E_{\mathcal{U}}$ and let $\delta > 0$ be given. With no loss of generality, we may pick representatives $(x_i), (y_i)$ of \mathbf{x} and \mathbf{y} , respectively, with $\|x_i\| = \|y_i\| = 1$ for each $i \in \Gamma$. (This is seemingly well-known; e.g. see [3, Lemma 3.1.1] for the proof.) Using the assumption (S), we choose $\mathbf{z} = [(z_i)]$ with $\|z_i\| \leq \delta$ ($i \in \Gamma$) such that for each $i \in \Gamma$ either

$$\|x_i + z_i\| > 1 \text{ \& } \|y_i + z_i\| < 1 \text{ or } \|x_i + z_i\| < 1 \text{ \& } \|y_i + z_i\| > 1.$$

The above conditions are mutually exclusive, so precisely one of the sets $Z_1 = \{i: \|x_i + z_i\| > 1 \text{ \& } \|y_i + z_i\| < 1\}$ or $Z_2 = \{i: \|x_i + z_i\| < 1 \text{ \& } \|y_i + z_i\| > 1\}$ belongs to \mathcal{U} . Thus, we may suppose that $Z_1 \in \mathcal{U}$.

We *claim* that $\|\mathbf{x} + \mathbf{z}\| > 1$, that is, $g = \lim_{i, \mathcal{U}} \|x_i + z_i\| > 1$. Assume not. This means that for each natural number n , there is $X_n \in \mathcal{U}$ such that $\|x_i + z_i\| - g < \frac{1}{n}$ for each $i \in X_n$.

In particular, as $\frac{1}{n}$ can be arbitrarily small, we must have $g = 1$. Since the ultrafilter \mathcal{U} is σ -complete, we conclude that $U = \bigcap_{n=1}^{\infty} U_n \in \mathcal{U}$. Manifestly, this is a contradiction because U must be empty and, consequently, $U \notin \mathcal{U}$. Indeed, in the opposite case, for all $i \in U$ we would have $\|x_i + z_i\| \leq 1$ since the inequality $\|x_i + z_i\| < 1 + \frac{1}{n}$ holds for each n . This contradicts the choice of z_i 's and completes the proof.

Arguing analogously, we deduce that $\|\mathbf{y} + \mathbf{z}\| < 1$. Since $\|\mathbf{z}\| \leq \delta$, we conclude that $E_{\mathcal{U}}$ has (S). \square

Theorem 9. *Assume there exists a measurable cardinal. Then there exists a strictly convex Banach space E , whose some ultrapower $E_{\mathcal{U}}$ has (S), nonetheless $E_{\mathcal{U}}$ cannot be renormed to a strictly convex space.*

Proof. Let κ be a measurable cardinal and let $A \cup B$ be a partition of κ into two disjoint sets with $|A| = \omega_1$. Consider the Banach space $E = c_0(\kappa) = c_0(A) \oplus c_0(B)$, renormed by Day's strictly convex norm (see [4]). Also, let $F = c_0(A)^{**} \oplus c_0(B)$. The space F is clearly isomorphic to $\ell_{\infty}(\omega_1) \oplus c_0(\kappa)$ and hence, cannot be renormed to a strictly convex space. (This is because F contains $\ell_{\infty}(\omega_1)$, which has no strictly convex renorming (*c.f.* [4]) and the property of being isomorphic to a strictly convex space is inherited by closed subspaces.) In the light of Proposition 8, it suffices then to embed F into $E_{\mathcal{U}}$, where \mathcal{U} is some σ -complete ultrafilter.

Let Γ be the set of all finite-dimensional subspaces of F . Denote by $[Z]^{<\omega}$ the family of all finite subsets of a given set Z . Then,

$$|\Gamma| = |[2^{\omega_1} + \kappa]^{<\omega}| = |[\kappa]^{<\omega}| = \kappa,$$

where the second equality follows from the strong inaccessibility of κ . Let \mathcal{U} be a κ -complete ultrafilter on Γ . Appealing to the Principle of Local Reflexivity (*e.g.* see [1, Theorem 11.2.4]), we infer that for each $M \in \Gamma$, $M \subset c_0(A)^{**}$, there exists a linear operator $T_M: M \rightarrow c_0(A)$ such that

$$\frac{1}{2}\|x\| \leq \|T_M x\| \leq \frac{3}{2}\|x\| \quad (x \in M).$$

This allows us to associate with every $M \in \Gamma$ a linear operator $S_M: M \rightarrow E$ given by the formula

$$S_M x = \begin{cases} T_M x, & x \in M \cap c_0(A)^{**} \\ x, & x \in M \cap c_0(B) \end{cases} \quad (M \in \Gamma).$$

Let us consider the ultraproduct $\prod_{M \in \Gamma}^{\mathcal{U}} S_M(F)$ as a subspace of $E_{\mathcal{U}}$. Define an operator $S: F \rightarrow \prod_{M \in \Gamma}^{\mathcal{U}} S_M(F) \subseteq E_{\mathcal{U}}$ by the formula

$$Sx = [(S_M x)_{M \in \Gamma}] \quad (x \in F).$$

We note that S is an isomorphic embedding. Indeed, S restricted to $c_0(B)$ is an isometry and, moreover, for all $x \in c_0(A)^{**}$ and $M \in \Gamma$ we have

$$\frac{1}{2}\|x\| \leq \|S_M x\| \leq \frac{3}{2}\|x\|.$$

Consequently, $\frac{1}{2}\|x\| \leq \lim_{M, \mathcal{U}} \|S_M x\| = \|Sx\| \leq \frac{3}{2}\|x\|$. \square

Remark 10. The space E we consider must have been chosen to have size at least κ . This is because for every Banach space X with cardinality less than κ , the ultrapower $X_{\mathcal{U}}$ is isometric to X , provided the ultrafilter \mathcal{U} is σ -complete.

Since Theorem 9 shows that, at least consistently, the property (S) does not imply the existence of an equivalent strictly convex norm, it is natural to ask whether some classical non-strictly convexifiable Banach spaces, like $\ell_{\infty}(I)$ for $|I| \geq \omega_1$ (c.f. [4] or [5, §4.5]) and ℓ_{∞}/c_0 (c.f. [2]), admit any equivalent norm satisfying (S).

REFERENCES

- [1] F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, Graduate Texts in Mathematics 233, Springer, 2006.
- [2] J. Bourgain, ℓ^{∞}/c_0 has no equivalent strictly convex norm, *Proc. Amer. Math. Soc.* **78** (1980), 225–226.
- [3] M. Daws, *Banach algebras of operators*. PhD. Thesis, University of Leeds, 2004.
- [4] M.M. Day, Strict convexity and smoothness, *Trans. Amer. Math. Soc.* **78** (1955), 516–528.
- [5] J. Diestel, *Geometry of Banach Spaces – Selected Topics*, Lecture Notes in Math. 485, Springer-Verlag, Berlin–Heidelberg–New York 1975.
- [6] D.H. Fremlin, *Measure Theory: Broad Foundation* (vol. 2), Torres Fremlin 2001.
- [7] T. Jech, *Set Theory*, Third Millennium Ed., revised and expanded, Springer-Verlag, Berlin, 2003.
- [8] S. Heinrich, Ultraproducts in Banach space theory. *J. Reine Angew. Math.* **313** (1980), 72–104.
- [9] N.Y. Luther, A decomposition of measures, *Canadian J. Math.* **20** (1968), 953–958.
- [10] D.S. Scott, Measurable cardinals and constructible sets. *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques* (continued from 1983 by *Bulletin of the Polish Academy of Sciences. Mathematics.*), **9** (1961), 521–524.
- [11] H. Steinhaus, *One Hundred Problems in Elementary Mathematics*, Dover Publ. 1965.
- [12] J. Stern, Some applications of model theory in Banach space theory, *Annals of Mathematical Logic*, **9**, no. 1-2 (1976), 49–121.
- [13] J. Stern, Ultrapowers and local properties of Banach spaces. *Trans. Amer. Math. Soc.* **240** (1978), 231–252.
- [14] P. Zwoleński, Some generalization of Steinhaus' lattice points problem, *Colloq. Math.* **123** (2011), 129–132.

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